

✓

6410
793
Cape
Lent

TECHNICAL MEMORANDUMS

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

No. 973

POTENTIAL FLOW THROUGH CENTRIFUGAL PUMPS AND TURBINES

By E. Sørensen

Zeitschrift für angewandte Mathematik und Mechanik
Vol. 7, No. 2, April 1927

Washington
April 1941



3 1176 01440 7218

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM NO. 973

POTENTIAL FLOW THROUGH CENTRIFUGAL PUMPS AND TURBINES*

By E. Sørensen

The methods of conformal transformation up to the present have been applied to the potential flows in the rotation of solid bodies only to a limited extent. The principal reason for this lies in the fact that the boundary conditions are not maintained after the transformation. All motions of solid bodies in the fluid lead to boundary-value problems of the second type, in the sense of the potential theory because on certain boundaries of a region, not the potential itself but its normal derivative, is given. In transformations referring to pure translations, this difficulty was met by passing from the absolute to the relative flow and thus converting the boundary-value problem of the second kind into one of the first kind. The method, however, is not applicable to motions of rotation since the relative motion is no longer free from vortices (irrotational). The case of rotational motion is, however, a very important one and absolutely necessary for investigations on centrifugal pumps and turbines. Some successful attempts have been made at treating this problem of rotational motion. Kucharski, in 1918, investigated the rotation of a radial-blade system, whose blades extended to the axis ("Flows of a Frictionless Fluid," Oldenbourg, 1918). Without making use of conformal transformation, he investigated the relative flow and computed it numerically with the aid of an approximation method. Spannhake, then, in 1925, using the absolute flow as a basis for his investigation, computed, with the aid of conformal transformation, the system of radial blades with arbitrary entrance and exit radii, and indicated the possibility of computing systems with arbitrary blade shapes (Z.f.a.M.M., Bd. 5, Heft 6, 1925; pp. 481-494). Both authors considered the so-called "two-dimensional flow," which will also be used as a basis of the present work.

*"Potentialströmungen durch rotierende Kreiselräder."
Z.f.a.M.M., Bd. 7, Heft 2, April, 1927, pp. 89-106.

1. THE COMPLEX POTENTIAL FOR THE ROTATION

The rotation of an arbitrarily shaped blade in a complex w plane gives rise to a definite and known normal component v_n of the velocity for all water particles on the blade surface. In the w plane let the blade be given by a piece of a curve $w_s = f_1(a)$, where a is a parameter that varies between definite limits $a_0 \leq a \leq a_1$. The angle α , at which the blade curve intersects, a ray from the origin of coordinates $w = 0$ (fig. 1) is similarly a function of a ; $\cos \alpha = f_2(a)$. Let the blade rotate with the angular velocity ω about the point $w = 0$. There then arises a peripheral velocity $u = \omega |w_s| = |f_1(a)| \omega$. The normal component of the velocity of a water particle on the blade surface is then $v_n = u \cos \alpha = \omega |f_1(a)| f_2(a)$. Now let the w plane, by a transformation $w = \zeta_1(z)$ be carried over into a z plane. The points in the z plane corresponding to the points w_s of the blade in the w plane again lie on a definite curve $z_s = \zeta_2(a)$ determined by the transformation function. It is now required to find the complex potential $\Phi(w)$, which in the w plane represents the rotation of the blade about the origin. This potential, after it has been transformed, must in the z plane at each point of the curve $z_s = \zeta_2(a)$, give a normal velocity

$$c_n = v_n \left| \frac{dw}{dz} \right|_{[z=z_s=\zeta_2(a)]}$$

$$c_n = \omega |f_1(a)| f_2(a) \left| \frac{d\zeta_1(z)}{dz} \right|_{[z=\zeta_2(a)]} = \zeta_3(a)$$

$c_n = \zeta_3(a)$ is similarly a definite and known function.

Now Spannhake chose for $z_s = \zeta_2(a)$ the equation of a circle at whose center he assumed a number of double sources (doublets) of higher order:

$$z_s = \zeta_2(a) = z_0 + q e^{ia}; \quad 0 \leq a \leq 2\pi, \quad \Phi(z) = \sum_{k=1}^{\infty} \frac{A_k + i B_k}{(z - z_0)^k}$$

Then

$$c_n = - \sum_{k=1}^{\infty} k [A_k \cos(ka) + B_k \sin(ka)] q^{-k-1} =$$

$$= \xi_3(a) = w \left| f_1(a) \right| f_2(a) \left| \frac{dw}{dz} \right|_{[z=\xi_2(a)]}$$

The potential is thus determined when the coefficients of the above Fourier series have been found. In this way it is fundamentally possible to find the complex potential for the flow through a rotating impeller with arbitrary number of blades and blade shape. The most important part of the work is the carrying through of a harmonic analysis - a process which can be carried out simply, only if the function $c_n = \xi_3(a)$ possesses a well converging Fourier series. This, however, is by no means always the case, and the example treated below by a different method indicates the practical impossibility of finding a sufficiently accurate series of this type for the required potential. The transformation of the blade $w = \xi_1(z)$ into a circle, can be made in an infinite number of ways and the function $c_n = \xi_3(a)$ depends again on the position of the center and on the radius of the transformed circle. It might be tried, therefore, to choose the center and radius in such a manner that $\xi_3(a)$ gives a strongly converging series; it cannot be predicted, however, that this will always lead to a successful solution.

The method chosen for the following computations gives the potential in the form - not of a series - but of a definite integral. (Lamb, Hydrodynamics sections 57, 58.) On the blade curve $z_s = \xi_2(a)$, a source and sink distribution is assumed. This source distribution can only be made if the blade curve $z_s = \xi_2(a)$ encloses a region in the z plane; i.e., if the top and bottom sides of the blade are separated. In the w plane, this is not necessary. The transformation function can be so chosen that the region enclosed by the blade contour in the z plane is transformed in the w plane into a second Riemann sheet. The blade will then appear in the w plane as a section in one Riemann sheet of the at-least two-sheet w plane, the blade tips appearing as branch points.

The quantity of water per second appearing or vanishing at a point of the curve $\xi_2(a)$, is similarly a function of a : $q(a) = 2\xi_3'(a)$. The quantity originating at a curve element ds is $dq = 2\xi_3'(a) ds$. The potential of

this elementary source is

$$d\Phi = \frac{dq}{2\pi} \ln(z - z_s) = \frac{1}{\pi} \xi_3'(a) ds \ln [z - \xi_2(a)] \quad (1)$$

$$ds = \left| \frac{d\xi_2(a)}{da} \right| da$$

The potential for the entire source distribution, therefore, is:

$$\Phi = \frac{1}{\pi} \int_{a_0}^{a_1} \xi_3'(a) \left| \frac{d\xi_2(a)}{da} \right| \ln [z - \xi_2(a)] da \quad (2)$$

$$\frac{d\Phi}{dz} = \frac{1}{\pi} \int_{a_0}^{a_1} \frac{\xi_3'(a) \left| \frac{d\xi_2(a)}{da} \right|}{z - \xi_2(a)} da = (c_n - ic_t) e^{-i\gamma} \quad (3)$$

where γ is the angle of inclination of the blade curve $z_s = \xi_2(a)$ to the axis of reals. The problem now is to find a source distribution which at each point gives the true required c_n component. In the general case, the difficulty by this method is thus only shifted - not overcome. There are cases, however, for which the source distributions can readily be indicated. One of these cases, for example, is that where the blade in the w plane can be transformed into a circle in the z plane; ($z_s = \xi_2(a)$ is then the equation of a circle). As is known, a source and sink of equal strengths separated by any distance, give rise to circles as streamlines. If a source and sink of equal strength can be arbitrarily located on the circle $z_s = \xi_2(a)$, then the latter becomes a streamline. The c_n component at any position then is not affected by any source or sink on the circle, with the exception of the one which lies at the position considered. The magnitude of c_n at any point, therefore, is proportional to the source strength at the point under consideration. The function for the source distribution $q(a) = 2\xi_3'(a)$ thus becomes identical with the function for the c_n component:

$$q(a) = 2c_n(a) \quad \text{or} \quad \xi_3'(a) = \xi_3(a)$$

In this manner the required source distribution and complex potential are found.

A second case is that considered in detail in what follows. The region about the blade in the w plane is transformed into the upper half-plane, and the region enclosed by the blade into the lower half-plane of the z plane. The blade contour becomes the axis of reals in the z plane. Each source or sink assumed on this axis makes the latter itself a streamline and thus gives rise to a c_n component only at the point where the source is located. The general parameter a , in this case is represented by the running coordinate of the axis of reals and in the following will be denoted by l ; also, in this case, $\xi_3(l) = \xi_5(l) = c_n(l)$. The function for the blade contour becomes:

$$\xi_3(l) = l; \quad -\infty \leq l \leq +\infty \quad (4)$$

Further, $ds = dl$

$$\Phi(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \xi_3(l) \ln(z - l) dl \quad (5)$$

$$\frac{d\Phi}{dz} = c_x - i c_y = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\xi_3(l) dl}{z - l} \quad (6)$$

This integral was first given by P. A. Walther (Transactions of the Central Aero-Hydrodynamical Institute, No. 18, Moscow, 1926).

The complex potential $\Phi(z)$ for each point $z = x + iy$ may be computed as the definite integral with respect to l as integration variable.

$$\frac{d\Phi}{dz} = c_x - i c_y = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\xi_3(l) [(x-l) - iy]}{(x-l)^2 + y^2} dl \quad (7)$$

Again it will be shown exactly that $\lim_{y \rightarrow 0} \left[\lim_{x \rightarrow l} [c_y(x,y)] \right] = \xi_3(l) = c_n(l)$. By separating, in equation (7), the real

and imaginary parts, there is obtained

$$c_y = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\xi_3(l) y dl}{(x-l)^2 + y^2} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{c_n(l) y dl}{(l-x)^2 + y^2} \quad (8)$$

Set $u \equiv (l - x)$; then $dl = du$. For $l = \pm \infty$, $u = \pm \infty$. Equation (8) now becomes

$$c_y = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{c_n(u+x) y du}{u^2 + y^2}$$

By splitting up into part integrals, there is obtained:

$$c_y = \frac{y}{\pi} \int_{-\infty}^{u_1} \frac{c_n(u+x) du}{u^2 + y^2} + \frac{1}{\pi} \int_{u_1}^{u_2} \frac{c_n(u+x) y du}{u^2 + y^2} + \frac{y}{\pi} \int_{u_2}^{+\infty} \frac{c_n(u+x) du}{u^2 + y^2} \quad (9)$$

where $u_1 < 0 < u_2$, y being considered as constant in

the integration. The integral $\Delta q = \int_{u_0}^{\infty} c_n(u) du$ gives the quantity of water originating on the piece of axis of reals $u > u_0$. This quantity is equal to that displaced by the corresponding piece of the rotating blade in the w plane and must therefore be finite. The integral can assume a finite value only if $[c_n(u)] = 0 = 1/\infty^k$ where k

is a number greater than 1. From this the result directly obtained is that the first and last integral of equation (9) must similarly be finite, because for $u \rightarrow \infty$ the integrand becomes infinitesimal to greater than the third order. Hence, $\lim [y/\pi (J_1 + J_3)] = 0$.

Since the function $F(u) = \frac{y}{y^2 + u^2}$ always has the sign of y , the integral J_2 can be transformed by the mean value theorem

$$\frac{1}{\pi} J_2 = \frac{1}{\pi} c_n (u_0 + x) \int_{u_1}^{u_2} \frac{y \, du}{u^2 + y^2}$$

where $u_0 = u_1 + \delta (u_2 - u_1)$ and $0 < \delta < 1$.

$$\frac{1}{\pi} J_2 = \frac{1}{\pi} c_n (u_0 + x) \left[(\tan^{-1}) \frac{u_2}{y} - (\tan^{-1}) \frac{u_1}{y} \right]$$

If now $y \rightarrow 0$, then $(\tan^{-1}) u_1/y \rightarrow -\pi/2$, because $u_1 < 0$, and $(\tan^{-1}) u_2/y \rightarrow +\pi/2$, because $u_2 > 0$. Hence,

$$\lim_{y \rightarrow 0} \left[\frac{1}{\pi} J_2 \right] = c_n [u_1 + \delta (u_2 - u_1) + x] = c_n (u_0 + x)$$

In order to determine at which position the value c_n of (u) should be taken, the two values u_1 and u_2 are made to approach zero. This does not change anything in the proof thus far. Since $u_1 < u_0 < u_2$, u_0 must also approach zero. There is thus obtained:

$$\lim_{y \rightarrow 0} \left[\lim_{\substack{u_1 \rightarrow 0 \\ u_2 \rightarrow 0}} \left(\frac{1}{\pi} \int_{u_1}^{u_2} \frac{c_n(u) y \, du}{u^2 + y^2} \right) \right] = c_n(x)$$

For $u = 0$ $x = l$, hence

$$\lim_{y \rightarrow 0} \left[\lim_{x \rightarrow l} (c_y(x, y)) \right] = c_n(l) \quad (10)$$

2. APPLICATION OF THE METHOD

To carry out practically the above-described procedure for the general flow through the rotating centrifugal pump, the flow is decomposed into two components. (See Spannake.) One component, the "discharge" flow, arises from the vortex source on the axis and the circulations about the blade for the impeller at start still. The blades are streamlines and this flow can be directly derived from

known potentials by a conformal transformation. The second part, the "displacement" flow, arises through the rotation of the impeller in the fluid at rest. The potential for this flow is given by the above-discussed integral.

In pump and turbine construction, overlapping blades - that is, those for which the angle $\psi > 2\pi/n$ (fig. 2) - are almost exclusively employed. Such an impeller, with blades whose contours are logarithmic spirals, will now be investigated in greater detail. For this purpose, the transformation that König has previously given (Z.f.a.M.M., Bd. 2, Heft 6, 1922), will be employed. This transformation is the following:

$$w_1 = e^{i\alpha} \ln \frac{z - z_a}{z - z_b} + e^{-i\alpha} \ln \frac{z - \bar{z}_a}{z - \bar{z}_b} \quad (11)$$

where $z_a = x_a + iy_a = r_a e^{i\theta_a}$ and $z_b = x_b + iy_b = r_b e^{i\theta_b}$ are two points on the positive half-plane of the z plane with distances r_a and r_b from the origin; \bar{z}_a and \bar{z}_b are the corresponding conjugate values (fig. 3). The given function transforms the real axis of the z plane into the "ladder" shown in figure 4. The "steps" are rectilinear and parallel to the axis of reals in the w plane, one step lying on the axis itself on the negative side, with one end at the origin. The length of a step is equal to $l = 2 [\cos \alpha \ln r_a/r_b - (\theta_a - \theta_b) \sin \alpha]$. The ladder axis, with the axis of imaginaries of the w plane, makes the angle α . $h = 2\pi \cos \alpha = t \cos \alpha$ is the vertical distance between the steps.

Now (see König, loc. cit.):

$$a) \quad \pi l/h = \ln \frac{\sin (\theta_b + \alpha - \pi/2)}{\sin (\theta_b + \pi/2 - \alpha)} - (\pi - 2\theta_b) \tan \alpha \quad (12)$$

$$b) \quad \theta_a + \theta_b = \pi \quad \dots \dots \dots (13)$$

$$c) \quad \frac{r_a}{r_b} = \frac{\sin (\theta_b + \alpha - \pi/2)}{\sin (\theta_b + \pi/2 - \alpha)} \quad \dots \dots \dots (14)$$

The transformation determines only the ratio r_a/r_b , but

not the values of r_a and r_b , themselves. One of the absolute values may be arbitrarily chosen; for example, $r_a = 1$. In this way the computations below are considerably simplified without restricting the generality of the results. The following relation now holds:

$$x_a^2 + y_a^2 = r_a^2 = 1 \text{ and further, since}$$

$$\theta_a = \pi - \theta_b, \quad x_a r_b = -x_b \quad (15)$$

$$y_a r_b = y_b \quad (16)$$

From the transformation equation, there is obtained:

$$w_1 = -\infty, \text{ if } z = z_a \text{ or } z = \bar{z}_a, \quad w_1 = 0, \text{ if } z = \infty,$$

$w_1 = +\infty$, if $z = z_b$ or $z = \bar{z}_b$. The image points of z_a , \bar{z}_a , z_b , \bar{z}_b thus move off toward infinity in the w_1 plane, while the image point of $z = \infty$ moves toward the origin.

By means of the transformation, $w_2 = w_1 e^{-i\alpha}$, the ladder is rotated by the angle α about the origin in the clockwise direction (fig. 5). The axis is now at right angles to the axis of reals with the steps inclined by the angle α .

By the transformation $w_3 = e^{w_2}$, a strip of the w_2 plane of width $2\pi i$ is transformed into a sheet of the w_3 plane ($w_3 = \rho_3 e^{i\psi_3}$). The points of all the steps now lie on a logarithmic spiral which intersects every ray through the origin at the angle α (fig. 6). The right end of the step corresponds to the point $\rho_3 = \rho_{3a} = 1$; $\psi_{3a} = 0$ in the w_3 plane, and the left end to the point $\rho_3 = \rho_{3l} = e^{-1} \cos \alpha$; $\psi_3 = \psi_{3l} = 1 \sin \alpha$ (in the diagram $\psi_{3l} > 2\pi$).

By the transformation $w = \frac{z}{\sqrt{w_3}}$, a piece of the spiral of the w_3 plane is transformed into n pieces of the logarithmic spiral in the w plane. These spirals

likewise intersect every ray through the origin at the angle α (fig. 2). To the point $\rho_{3a} = 1$, $\psi_{3a} = 0$ correspond the n points $\sqrt[n]{1} = 1 e^{ik \frac{2\pi}{n}}$; $k = 0, 1, 2$ to $n - 1$. The inner blade (spiral) ends lie on the circle with $\rho_1 = \sqrt[n]{\rho_{31}}$. The angle ψ subtended by a blade is equal to $\frac{\psi_{31}}{n}$; the blade spacing angle is equal to $2\pi/n$. The ratio of these two angles is denoted by m (fig. 5).

$$m = \frac{\psi}{2\pi} = \frac{\psi_{31}}{2\pi} = \frac{l \sin \alpha}{t} = l \frac{\sin \alpha \cos \alpha}{h} = \frac{l \sin 2\alpha}{h} \quad (17)$$

The successive transformation functions can be combined into a single one. From

$$w_1 = e^{i\alpha} \ln \frac{z - z_a}{z - z_b} + e^{-i\alpha} \ln \frac{z - \bar{z}_a}{z - \bar{z}_b}; \quad w_2 = w_1 e^{-i\alpha};$$

$$w_3 = e^{w_2}; \quad w = \sqrt[n]{w_3} \quad \text{follows:}$$

$$w = \left[\left(\frac{z - z_a}{z - z_b} \right)^{e^{i\alpha}} \left(\frac{z - \bar{z}_a}{z - \bar{z}_b} \right)^{e^{-i\alpha}} \right]^{\frac{1}{n}} e^{-i\alpha} \quad (18)$$

This is the function that transforms the real axis of the z plane into a blade system with n blades in the w plane. In this function α , z_a , and z_b are still undetermined; α is the angle formed by the blade with the radius vector, and z_a and z_b are determined from equations (12), (13), and (14) if a definite ratio of radii or overlap ratio m is prescribed. From n there is first determined by equation (17) l/h ; then θ_b by equation (12), and finally, r_a/r_b by equation (14).

3. POTENTIAL FOR THE FLOW WITH IMPELLER AT REST

The transformation equation (18) shows that for $z = z_a$ $w = 0$ and for $z = z_b$ $w = \infty$. For the flow with blades at rest, the point $w = 0$ must be a vortex source and the point $w = \infty$, a vortex sink. At the point z_a of the z plane there must therefore be located a vortex source,

and at the point z_b a vortex sink. In order that the axis of reals shall be a streamline, the source and sink must be reflected in the points \bar{z}_a and \bar{z}_b . The potential for the flow in the z plane, therefore, is:

$$\begin{aligned} \Phi_d(z) = & (q + im) \ln (z - z_a) + (s + io) \ln (z - z_b) \\ & + (q - im) \ln (z - \bar{z}_a) + (s - io) \ln (z - \bar{z}_b) \end{aligned} \quad (19)$$

(See Kóniz, loc. cit.), where q is the strength of the sources, s that of the sinks, and m and o are the vortex strengths. The continuity is satisfied only if $q = -s$ because there must be no sink at infinity.

4. POTENTIAL FOR THE FLOW DUE TO THE BLADE ROTATION

The rotation of the impeller in the w plane gives rise at the blade surface to normal components of the water velocity $v_n = |\omega| w \cos \alpha$. By the function (18) the blades of the w plane are transformed into the axis of reals of the z plane. Each point $z = l$ of this axis thus corresponds to a blade point w , the value of l varying from $-\infty$ to $+\infty$. A potential must now be found which at each point l gives rise to a normal velocity:

$$\begin{aligned} c_n &= v_n \left| \frac{dw}{dz} \right|_{(z=l)} = \omega |w_s| \cos \alpha \left| \frac{dw}{dz} \right|_{(z=l)} \\ w &= \left[\left(\frac{z - z_a}{z - z_b} \right)^{e^{i\alpha}} \left(\frac{z - \bar{z}_a}{z - \bar{z}_b} \right)^{e^{-i\alpha}} \right]^{\frac{1}{n}} e^{-i\alpha} \\ \frac{dw}{dz} &= \frac{1}{n} e^{-i\alpha} \left[\left(\frac{z - z_a}{z - z_b} \right)^{e^{i\alpha}} \left(\frac{z - \bar{z}_a}{z - \bar{z}_b} \right)^{e^{-i\alpha}} \right]^{\frac{1}{n} - 1} e^{-i\alpha - 1} \\ & \left[e^{i\alpha} \left(\frac{z - \bar{z}_a}{z - \bar{z}_b} \right)^{e^{-i\alpha}} \left(\frac{z - z_a}{z - z_b} \right)^{e^{i\alpha} - 1} \frac{z_a - z_b}{(z - z_b)^2} + e^{-i\alpha} \left(\frac{z - z_a}{z - z_b} \right)^{e^{i\alpha}} \left(\frac{z - \bar{z}_a}{z - \bar{z}_b} \right)^{e^{-i\alpha} - 1} \frac{\bar{z}_a - \bar{z}_b}{(z - \bar{z}_b)^2} \right] \\ &= \frac{1}{n} e^{-i\alpha} \left[\left(\frac{z - z_a}{z - z_b} \right)^{e^{i\alpha}} \left(\frac{z - \bar{z}_a}{z - \bar{z}_b} \right)^{e^{-i\alpha}} \right]^{\frac{1}{n} - 1} e^{-i\alpha} \left[\frac{(z_a - z_b) e^{i\alpha}}{(z - z_a)(z - z_b)} + \frac{(\bar{z}_a - \bar{z}_b) e^{-i\alpha}}{(z - \bar{z}_a)(z - \bar{z}_b)} \right] \end{aligned} \quad (20)$$

For real $z = l$, the magnitudes in the two brackets are themselves also real, because in this case they consist of a product or a sum of conjugate complex numbers. Moreover $e^{-i\alpha} = 1$. The absolute value of dw/dz , on substituting the real value l for z , can therefore be written as follows:

$$\left| \frac{dw}{dz} \right|_{(z=l)} = \frac{1}{n} \left| \left[\left(\frac{l - z_a}{l - z_b} \right)^{e^{i\alpha}} \left(\frac{l - \bar{z}_a}{l - \bar{z}_b} \right)^{e^{-i\alpha}} \right] e^{-i\alpha \frac{1}{n}} \right|$$

$$\times \left[\frac{(z_a - z_b) e^{i\alpha}}{(l - z_a)(l - z_b)} + \frac{(\bar{z}_a - \bar{z}_b) e^{-i\alpha}}{(l - \bar{z}_a)(l - \bar{z}_b)} \right] \quad (21)$$

Hence at the blade:

$$c_n = \frac{1}{n} w \cos \alpha \left| \left[\left(\frac{l - z_a}{l - z_b} \right)^{e^{i\alpha}} \left(\frac{l - \bar{z}_a}{l - \bar{z}_b} \right)^{e^{-i\alpha}} \right]^{\frac{2}{n}} e^{-i\alpha} \right|$$

$$\times \left[\frac{(z_a - z_b) e^{i\alpha}}{(l - z_a)(l - z_b)} + \frac{(\bar{z}_a - \bar{z}_b) e^{-i\alpha}}{(l - \bar{z}_a)(l - \bar{z}_b)} \right] \quad (22)$$

The following notation is now introduced:

$$\frac{l - z_a}{l - z_b} = \mu e^{i\nu} \quad \text{and} \quad \frac{l - \bar{z}_a}{l - \bar{z}_b} = \mu e^{-i\nu} \quad (23)$$

$$\frac{l - z_a}{l - z_b} = \frac{(l - x_a)(l - \bar{z}_b)}{(l - z_b)(l - \bar{z}_b)} = \frac{(l - x_a - iy_a)(l - x_b + iy_b)}{(l - x_b - iy_b)(l - x_b + iy_b)}$$

$$= \frac{(l - x_a)(l - x_b) + y_a y_b + i(l - x_a)y_b - (l - x_b)y_a}{(l - x_b)^2 + y_b^2} = \mu e^{i\nu}$$

Hence the absolute value and the argument are:

$$\mu = \frac{\sqrt{[(l - x_a)(l - x_b) + y_a y_b]^2 + [(l - x_a)y_b - (l - x_b)y_a]^2}}{(l - x_b)^2 + y_b^2} = \mu(l) \quad (24)$$

$$\tan v = \frac{(1 - x_a) y_b - (1 - x_b) y_a}{(1 - x_a)(1 - x_b) + y_a y_b}; \quad v = v(l) \quad (25)$$

$$\left(\frac{1 - z_a}{1 - z_b}\right)^{e^{i\alpha}} = (\mu e^{iv}) e^{i\alpha} = \mu \cos \alpha e^{-v \sin \alpha} e^{i(\ln \mu \sin \alpha + v \cos \alpha)}$$

$$\left(\frac{1 - \bar{z}_a}{1 - \bar{z}_b}\right)^{e^{-i\alpha}} = \mu \cos \alpha e^{-v \sin \alpha} e^{-i(\ln \mu \sin \alpha + v \cos \alpha)}$$

Hence

$$\left(\frac{1 - z_a}{1 - z_b}\right)^{e^{i\alpha}} \left(\frac{1 - \bar{z}_a}{1 - \bar{z}_b}\right)^{e^{-i\alpha}} = \mu^2 \cos \alpha e^{-2v \sin \alpha} \quad (26)$$

Now raise the expression (26) to the power $\frac{n}{2} e^{-i\alpha}$

$$[\mu^2 \cos \alpha e^{-2v \sin \alpha}]^{n/2} (\cos \alpha - i \sin \alpha) =$$

$$= \mu^{n/2} (1 + \cos 2\alpha) e^{-n/2 v \sin 2\alpha} e^{i n/2 (\ln \mu \sin 2\alpha - 2v \sin^2 \alpha)}$$

The absolute value is therefore

$$\left| \left[\left(\frac{1 - z_a}{1 - z_b}\right)^{e^{i\alpha}} \left(\frac{1 - \bar{z}_a}{1 - \bar{z}_b}\right)^{e^{-i\alpha}} \right]^{n/2} e^{-i\alpha} \right| = \mu^{n/2} (1 + \cos 2\alpha) e^{-n/2 v \sin 2\alpha} = M(l)^{1/n} \quad (27)$$

Now transform the last factor of (22):

$$\begin{aligned} & \frac{(z_a - z_b) e^{i\alpha}}{(1 - z_a)(1 - z_b)} + \frac{(\bar{z}_a - \bar{z}_b) e^{-i\alpha}}{(1 - \bar{z}_a)(1 - \bar{z}_b)} = \\ & = \frac{(z_a - z_b)(1 - \bar{z}_a)(1 - \bar{z}_b) e^{i\alpha} + (\bar{z}_a - \bar{z}_b)(1 - z_a)(1 - z_b) e^{i\alpha}}{(1 - z_a)(1 - z_b)(1 - \bar{z}_a)(1 - \bar{z}_b)} \end{aligned}$$

This expression is also given in Kori's paper (loc. cit. (equation (5))), where the condition is expressed that

the coefficient of l^3 and the term not containing l in the numerator, must be equal to zero. This condition is satisfied through the equations (12), (13), (14). Taking this condition into account, there is obtained by multiplying and collecting terms:

$$\frac{(z_a - z_b)e^{i\alpha}}{(l - z_a)(l - z_b)} + \frac{(\bar{z}_a - \bar{z}_b)e^{-i\alpha}}{(l - \bar{z}_a)(l - \bar{z}_b)} =$$

$$= \frac{2l [\cos \alpha (x_b^2 - x_a^2 + y_b^2 - y_a^2) - 2 \sin \alpha (x_a y_b - x_b y_a)]}{[(l - x_a)^2 + y_a^2][(l - x_b)^2 + y_b^2]} \quad (28)$$

Thus c_n is determined as a real function of l :

$$c_n(l) = \frac{1}{n} \omega \cos \alpha \mu(l)^{2/n(1+\cos \alpha)} e^{2/n \nu \sin \alpha}$$

$$\times \frac{k l}{[(l - x_a)^2 + y_a^2][(l - x_b)^2 + y_b^2]} \quad (29)$$

$$k = 2 [\cos \alpha (x_b^2 - x_a^2 + y_b^2 - y_a^2) - 2 \sin \alpha (x_a y_b - x_b y_a)] \quad (30)$$

k is thus independent of l while $\mu(l)$ and $\nu(l)$ have the values given by equations (24) and (25), the absolute value of $\mu(l)$ being always greater than zero. $\lim_{l \rightarrow \infty} \mu(l)$

$= 1$ because numerator and denominator are of the same degree and the highest powers of l in the numerator and denominator have the factor 1. $\nu(l)$ fluctuates between the limits $\nu_{\max} > 0$ and $\nu_{\min} < 0$; $\lim_{l \rightarrow \infty} \nu(l) = 0$

because the denominator is of higher degree than the numerator. $M(l) = \mu(l)^{2(1+\cos \alpha)} e^{-2\nu(l)\sin \alpha}$ is always positive and finally

$$\lim_{l \rightarrow \infty} M(l) = 1 \quad (31)$$

$$\lim_{n \rightarrow \infty} [M(l)^{1/n}] = 1 \quad (32)$$

$c_n(l)$ becomes zero to the first order if $l \rightarrow 0$.

Moreover, $[c_n(l)] = 0$ to the third order because, according to equation (29) the denominator of the last factor is three degrees higher than the numerator.

The potential for the displacement flow in the z plane, according to equation (5), is:

$$\Phi_v(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} c_n(l) \ln(z-l) dl$$

where $c_n(l)$ has the above-determined meaning. The complete potential for the combined flow components is equal to the sum of the individual potentials:

$$\Phi(z) = \Phi_d(z) + \Phi_v(z) \quad (33)$$

The corresponding transformation functions are given by equation (18).

5. VELOCITIES AT THE BLADE TIPS

In the complex potential function (33), there are involved a number of magnitudes whose choice provides the possibility of representing the most varied types of flow (ω , p , σ , n). Mathematically, these magnitudes may be chosen arbitrarily without violating the single condition that so far has been imposed. This is the continuity condition, which is expressed by the Laplace differential equation:

$$\Delta \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

A solution of this equation is given by the complex potential (33). Practically, that is, physically, two further conditions still exist which thus exclude the possibility of a completely arbitrary choice of all these magnitudes. The one condition, which practically always may be assumed as satisfied, states that the water always flows off tangentially from the impeller blades. The other condition is generally not satisfied in practice, though it is attempted

always to satisfy it as nearly as possible. This is on the condition that the water approaches the impeller blades tangentially or, as the pump designer expresses it, that entrance be impact-free. With both conditions simultaneously satisfied, the pump attains approximately its optimum efficiency, and is said to work in its normal operating condition. This condition will be assumed as the basis for further investigations. A large majority of pumps have no guide vanes ahead of the impeller; therefore the water enters the impeller without internal circulation. This would mean that m must be set equal to zero. This, however, will not be assumed to be the case at first, in order not to restrict the generality of results too greatly. Mathematically expressed, the above physical conditions state that the derivative of the potential $\Phi(w)$ in the w plane at the points corresponding to the blade tips, has a finite value. The nontangential flow is thus always (mathematically!) associated with the occurrence of infinitely large velocities. We must therefore have $b \leq$

$$\left| \frac{d\Phi}{dw} \right|_{\substack{w=w_{si} \\ w=w_{sa}}} \leq B \quad \text{where } b \text{ and } B \text{ are two finite magnitudes and}$$

w_{si} and w_{sa} denote respectively the inner and the outer tips of the blade. In the z plane these points correspond to the points $z = 0$ and $z = \infty$, respectively.

Now, $\frac{d\Phi(w)}{dw} = \frac{d\Phi(z)}{dz} \frac{dz}{dw}$; hence,

$$b \leq \left| \frac{d\Phi(z)}{dz} \frac{dz}{dw} \right|_{\substack{z=0 \\ z=\infty}} \leq B$$

$$\frac{dz}{dw} = n e^{i\alpha} \left[\left(\frac{z-z_a}{z-z_b} \right)^{e^{i\alpha}} \left(\frac{z-\bar{z}_a}{z-\bar{z}_b} \right)^{e^{-i\alpha}} \right]^{-1/n} e^{-i\alpha} \\ \times \frac{(z-z_a)(z-z_b)(z-\bar{z}_a)(z-\bar{z}_b)}{k z}$$

with account taken of equations (20), (28), and (30).

$$\left| \frac{dz}{dw} \right|_{z=0} \quad \text{thus becomes infinite to the first order, be-}$$

cause the first factors and the numerator of the last factor remain finite while the denominator of the last fac-

tor approaches zero linearly. In order that $\frac{d\Phi(w)}{dw}$ have a finite value, $\frac{d\Phi(z)}{dz}$ must equal zero to the first order if z is set equal to zero. The first condition equation (for impact-free entrance) thus reads:

$$\left[\frac{d\Phi_d(z)}{dz} + \frac{d\Phi_v(z)}{dz} \right]_{z=0} = 0$$

Now

$$\begin{aligned} \frac{d\Phi_d(z)}{dz} &= \frac{q + im}{z - z_a} + \frac{s + io}{z - z_b} + \frac{q - im}{z - \bar{z}_a} + \frac{s - io}{z - \bar{z}_b} \\ \left[\frac{d\Phi_d(z)}{dz} \right]_{z=0} &= - \left(\frac{q + im}{z_a} + \frac{s + io}{z_b} + \frac{q - im}{\bar{z}_a} + \frac{s - io}{\bar{z}_b} \right) \\ &= - 2 \left(\frac{qx_a - my_a}{x_a^2 + y_a^2} + \frac{sx_b - oy_b}{x_b^2 + y_b^2} \right) \end{aligned}$$

The complex potential for the displacement flow is:

$$\Phi_v(z) = \int_{-\infty}^{+\infty} F(l, z) dl, \quad \text{where } F(l, z) = c_n(l) \ln(z-l)$$

The function $F(l, z)$ satisfies the following conditions:

1. $F(l, z)$ and the partial derivative $\frac{\partial F(l, z)}{\partial z} = \frac{c_n(l)}{z - l}$ are continuous for every arbitrary l and every $z \neq l$.
2. $F(l, z)$ and $\frac{\partial F(l, z)}{\partial z}$ become infinitesimal to higher than first order if $l \rightarrow \pm \infty$. The derivative can therefore be obtained by differentiating under the integral sign:

$$\frac{d\Phi_v(z)}{dz} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{c_n(l) dl}{z - l}$$

$$\left[\frac{d\Phi_v(z)}{dz} \right]_{z=0} = - \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{c_n(l)}{l} dl = - \frac{1}{\pi} N_2 \quad (34)$$

with

$$N_2 = \int_{-\infty}^{+\infty} \frac{c_n(l)}{l} dl \quad (35)$$

so that the condition equation assumes the form

$$\left[\frac{d\Phi(z)}{dz} \right]_{z=0} = - 2 \left(\frac{qx_a - my_a}{1} + \frac{sx_b - oy_b}{r_b^2} \right) - \frac{1}{\pi} N_2 = 0 \quad (36)$$

In order to be able to set up the condition for the tangential flow, the derivatives $\frac{d\Phi_d(z)}{dz}$ and $\frac{d\Phi_v(z)}{dz}$ must be expanded in series of decreasing powers of z

$$\begin{aligned} \frac{d\Phi_d(z)}{dz} &= \frac{(q+im)(z-\bar{z}_a) + (q-im)(z-z_a)}{(z-z_a)(z-\bar{z}_a)} \\ &\quad + \frac{(s+io)(z-\bar{z}_b) + (s-io)(z-z_b)}{(z-z_b)(z-\bar{z}_b)} \end{aligned}$$

By dividing out there is obtained:

$$\frac{d\Phi_d(z)}{dz} = \frac{2}{z} (q+s) + \frac{2}{z^2} (qx_a - my_a + sx_b - oy_b) + \frac{2}{z^3} (\dots) + \dots (37)$$

$$\begin{aligned} \frac{d\Phi_v(z)}{dz} &= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{c_n(l) dl}{z-l} = \frac{1}{\pi} \left\{ \frac{1}{z} \int_{-\infty}^{+\infty} c_n(l) dl \right. \\ &\quad \left. + \frac{1}{z^2} \int_{-\infty}^{+\infty} c_n(l) l dl + \frac{1}{z^3} \int_{-\infty}^{+\infty} \frac{c_n(l) l^2 dl}{z-l} \right\} \quad (38) \end{aligned}$$

since

$$\frac{1}{z-l} = \frac{1}{z} + \frac{l}{z^2} + \frac{l^2}{z^3} + \frac{1}{z-l}$$

It will now be shown that the integral $\int_{-\infty}^{+\infty} \frac{c_n(l) l^2 dl}{z - l}$

becomes equal to zero if $z \rightarrow \infty$. For this purpose, $c_n(l) l^2 = f(l)$ must be developed into a power series in $(1/l)$.

$$c_n(l) l^2 = \alpha \mu(l)^b e^{c\nu(l)} \frac{l^3}{l^4 + dl^3 + el^2 + fl + g}$$

The three factors, functions of l , may all be represented as power series in $(1/l)$ which converge absolutely within a definite interval $-r_1 < 1/l < r_2$, which also includes the point $1/l = 0$. Also the product of these three factors may then be represented as a power series for $(1/l)$ with a definite convergence interval. As shown above, $\lim_{l \rightarrow \infty} (c_n(l) l^2) = 0$ to the first order.

The power series thus has the following form:

$$c_n(l) l^2 = \frac{a_1}{l} + \frac{a_2}{l^2} + \frac{a_3}{l^3} + \dots = \sum_{n=1}^{\infty} \frac{a_n}{l^n} \quad (39)$$

and converges in the interval $\rho_1 = 1/l_1' < 1/l < 1/l_2' = \rho_2$ where $l_1' < 0$ and $l_2' > 0$. For all values within the convergence interval the series converges absolutely; if $l < l_1'$ or $l > l_2'$. The limits of the convergence intervals are not of importance, the essential consideration being that such an interval with finite limits l_1' , l_2' exists.

The integral to be investigated is split up into three-part integrals:

$$\int_{-\infty}^{+\infty} \frac{c_n(l) l^2 dl}{z - l} = \int_{-\infty}^{l_1} \frac{c_n(l) l^2 dl}{z - l} + \int_{l_1}^{l_2} \frac{c_n(l) l^2 dl}{z - l} + \int_{l_2}^{+\infty} \frac{c_n(l) l^2 dl}{z - l}$$

$$J = J_1 + J_2 + J_3$$

$l_1 < l_1' < 0$ and $l_2 > l_2' > 0$. The integral J_2 converges to zero for increasing z since it has finite limits and $\lim_{z \rightarrow \infty} (z - l) = \infty$; $\lim J_2 = 0$ to the first order.

In order to estimate the integrals J_1 and J_3 , the expression $\frac{c_n(l)l^2}{z - l}$ is transformed by setting for $c_n(l)l^2$ the power series:

$$\frac{c_n(l)l^2}{z - l} = \frac{a_1}{l(z-l)} + \frac{a_2}{l^2(z-l)} + \frac{a_3}{l^3(z-l)} + \dots$$

The general term $\frac{1}{l^n(z-l)}$ is broken up into a term with $z - l$ in the denominator, and the remainder developed into a series of powers in l .

$$\frac{1}{l^n(z-l)} = \frac{1}{z^n(z-l)} + \frac{1}{z^n l} + \frac{1}{z^{n-1} l^2} + \dots + \frac{1}{z l^n}$$

Then

$$\frac{c_n(l)l^2}{z - l} = \sum_{n=1}^{\infty} a_n \left\{ \frac{1}{z^n(z-l)} + \frac{1}{z^n l} + \dots + \frac{1}{z l^n} \right\}$$

Now

$$\begin{aligned} \frac{1}{|l|^n (|z| - |l|)} &= \frac{1}{|z|^n (|z| - |l|)} + \frac{1}{|z|^n |l|} \\ &+ \frac{1}{|z|^{n-1} |l|^2} + \dots + \frac{1}{|z| |l|^n} \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{|z|^n |z-l|} &= \frac{1}{|z|^n (|z| - |l|)} + \frac{1}{|l|^n (|z| - |l|)} = \frac{1}{|z|^n |z-l|} \\ &+ \frac{1}{|z|^n |l|} + \frac{1}{|z|^{n-1} |l|^2} + \dots + \frac{1}{|z| |l|^n} \end{aligned}$$

Multiplying by $|a_n|$ and adding, gives:

$$\left(\frac{1}{z-l} - \frac{1}{|z|-|l|} \right) \sum_{n=1}^{\infty} \frac{|a_n|}{|z|^n} + \frac{1}{(|z|-|l|)^n} \sum_{n=1}^{\infty} \frac{|a_n|}{|l|^n}$$

$$= \sum_{n=1}^{\infty} |a_n| \left\{ \frac{1}{|z|^n |z-l|} + \frac{1}{|z|^n |l|} + \dots + \frac{1}{|z| |l|^n} \right\}$$

The two series on the left side converge absolutely, however, only if z and l fall within the interval of convergence of the series (39) $c_n(l)l^n = \sum_{n=1}^{\infty} \frac{a_n}{l^n}$. Hence the series on the right also converges absolutely and thus satisfies the condition for change in order of the terms of the series. By a disarrangement of the terms, there is obtained:

$$\frac{c_n(l)l^n}{z-l} = \sum_{n=1}^{\infty} a_n \left\{ \frac{1}{z^n(z-l)} + \frac{1}{z^n l} + \dots + \frac{1}{z l^n} \right\}$$

$$= \left(\frac{1}{z-l} - \frac{1}{l} \right) \sum_{n=1}^{\infty} \frac{a_n}{z^n} + \sum_{\lambda=1}^{\infty} \frac{1}{z^\lambda} \sum_{r=1}^{\infty} \frac{a_{\lambda+r}}{l^{r+1}}$$

Now

$$\sum_{\lambda=1}^{\infty} \frac{1}{z^\lambda} \sum_{r=1}^{\infty} \frac{a_{\lambda+r}}{l^{r+1}} = \frac{1}{z} \frac{1}{l^2} \sum_{\lambda=1}^{\infty} \frac{1}{z^{\lambda-1}} \sum_{r=1}^{\infty} \frac{a_{\lambda+1}}{l^{r-1}}$$

This double series starts with a term that contains $1/l^2$; therefore it can be integrated from $-\infty$ to $l_1 < 0$, and from $l_2 > 0$ to $+\infty$ if l_1 and l_2 lie in the interval of convergence. The value of these integrals approaches zero as $z \rightarrow \infty$.* There still remain to be con-

sidered in greater detail, the two integrals $s_1 \int_{-\infty}^{l_1} \left(\frac{1}{z-l} + \frac{1}{l} \right) dl$ and $s_2 \int_{l_2}^{+\infty} \left(\frac{1}{z-l} + \frac{1}{l} \right) dl$ ($s_1 = \sum_{n=1}^{\infty} \frac{a_n}{z^n}$). The

following expression will first be computed:

* For this convergence proof as well as various suggestions for the mathematical treatment, the author is indebted to Professor Boehm, Karlsruhe.

$$\begin{aligned}
 s_1 \int_{l_u}^{l_1} \frac{dl}{z-l} + s_1 \int_{l_2}^{l_0} \frac{dl}{z-l} &= -s_1 \ln \frac{z-l_1}{z-l_u} - s_1 \ln \frac{z-l_0}{z-l_2} \\
 &= -s_1 \ln \frac{(z-l_1)(z-l_0)}{(z-l_u)(z-l_2)}
 \end{aligned}$$

As is known, the limiting value of such expressions as $l_0 \rightarrow +\infty$ and $l_u \rightarrow \infty$ depends on the manner in which the limit is approached. The result will be different, for example, if we set $l_0 = l_u$ or $l_0 = l_u^2$, and then pass to the limit. In this case, however, the manner of approaching the limit is given by the type of transformation function $w = f(z)$, l_0 and l_u being the two values in the z plane which correspond to the same point of the blade in the w plane. If, now, w moves toward the end of the blade turned away from the origin, then l_0 and l_u approach infinity in such a manner that $|l_0/l_u| = 1$. (See equations (22) and (23).)

In the expression

$$\lim_{z \rightarrow \infty} \left[\lim_{\substack{l_0 \rightarrow +\infty \\ l_u \rightarrow -\infty}} \left[\ln \frac{(z-l_1)(z-l_0)}{(z-l_u)(z-l_2)} \right] \right]$$

we must therefore set $|l_0| = |l_u|$. There is then obtained

$$\lim_{z \rightarrow \infty} \left[\lim_{\substack{l_0 \rightarrow +\infty \\ l_u \rightarrow -\infty}} \left[\ln \frac{(z-l_1)(z-l_0)}{(z-l_u)(z-l_2)} \right] \right] = \lim_{z \rightarrow \infty} \left[\ln \frac{z-l_1}{z-l_2} \right] = 0$$

Similarly, there is found:

$$\int_{-\infty}^{l_1} \frac{a_1 dl}{l} + \int_{l_2}^{+\infty} \frac{a_1 dl}{l} = a_1 \lim_{\substack{l_0 \rightarrow +\infty \\ l_u \rightarrow -\infty}} \left[\ln \frac{l_1 l_0}{l_u l_2} \right] = a_1 \ln \frac{l_1}{l_2}$$

By combining the two partial results, there is obtained:

$$\lim_{z \rightarrow \infty} \left[\int_{-\infty}^{+\infty} \frac{c_n(l) l^2 dl}{z - l} \right] = 0 \quad (40)$$

The two derivatives $\frac{d\phi_d(z)}{dz}$ and $\frac{d\phi_T(z)}{nz}$ are now expanded in decreasing powers of z . For $z \rightarrow \infty$, the derivatives become equal to zero to the third order. This is made possible by the fact that the coefficients of the terms with $1/z$ and $1/z^2$ are set equal to zero. In this way there arise two further equations of condition:

$$2(q+s) + \frac{1}{\pi} \int_{-\infty}^{+\infty} c_n(l) dl = 0 \quad (41)$$

$$2(qx_a - my_a + sx_b - oy_b) + \frac{N_1}{\pi} = 0 \quad (42)$$

where there is introduced the relation

$$\int_{-\infty}^{+\infty} c_n(l) l dl = N_1 \quad (43)$$

This is the condition for tangential exit. The first of these equations merely states that there is no divergence. The integral $\int_{-\infty}^{+\infty} c_n(l) dl$ gives the entire mass of water in the z plane which the impeller blade, considered as a source, would produce. This quantity, naturally, must be zero since the blade is actually free of any sources. Accordingly,

$$q = -s \quad (44)$$

In order to bring out the angular velocity in the remaining two conditions the normal component, which is produced at the angular velocity $\omega = 1$ is denoted by $c_n'(l)$, so that

$$c_n(l) = \omega c_n'(l) \quad (45)$$

Moreover, the angular velocity and the circulation about the blades with impact-free entrance will be denoted by ω_0' and Γ_0' , respectively. The two main equations then become (with $N_{1,2} = \omega_0' N_{1,2}'$)

$$2 \left(qx_a - my_a - \frac{qx_b + \Gamma_0' y_b}{r_b} \right) + \frac{\omega_0'}{\pi} m_2' = 0 \quad (46)$$

$$2 (qx_a - my_a - qx_b - \Gamma_0' y_b) + \frac{\omega_0'}{\pi} N_1' = 0 \quad (47)$$

so that in both condition equations, only the terms $qx_a - my_a$, $qx_b + \Gamma_0' y_b$ and r_b occur.

6. COMPUTATION OF THE PERFORMANCE

The hydraulic torque developed by the pump is $D = \frac{Q\gamma}{g} \frac{\Sigma \Gamma_s}{2\pi}$, the delivery head produced is $H = \frac{\omega}{g} \frac{\Sigma \Gamma_s}{2\pi}$ - where ω is the angular velocity, Q the quantity of water, and Γ_s the circulation about a blade. The flow in the impeller is congruent, however, in the fields between two blades, so that $\Sigma \Gamma_s = n \Gamma_s$ where n is the number of blades. In the notation thus far employed, $2\pi m$ is the circulation on a curve within the impeller-blade system about the axis of the impeller, while $2\pi o$ is the circulation of the water discharged from the impeller. It is to be noted, however, that with $w = \frac{n}{\sqrt{3}}$, the values q, m, o are multiplied by n . In the w plane, therefore, the water discharge $Q = 2\pi n q$, inner circulation $\Gamma_i = 2\pi n m$, outer circulation $\Gamma_o = 2\pi n o$, circulation about the blade $\Gamma_s = 2\pi(o-m)$. The torque of the centrifugal pump investigated is thus $D = \frac{2\pi q n^2}{g} (o - m)$ and the delivery head $H = \frac{\omega}{g} n (o - m)$. Equations (46) and (47) now make possible the computation of the torque and delivery head for each speed and rate of discharge. For the following computations, there will be assumed a pump without guide vanes at the entrance; i.e., with water approach free from any rotational component $n = 0$. Equations (46) and (47) then become:

$$q(x_a r_b^2 - x_b) - O_o' y_b + \frac{\omega_o'}{2\pi} r_b^2 N_2' = 0$$

$$q(x_a - x_b) - O_o' y_b + \frac{\omega_o'}{2\pi} N_1' = 0$$

From these and equations (15) and (16) follows:

$$\omega_o' = \frac{2\pi (x_b r_b + x_a)}{-N_1' + r_b N_2'} q \quad (48)$$

$$O_o' = q \frac{(x_b - x_a) (N_1' - r_b N_2')}{(-N_1' + r_b^2 N_2') y_a} \quad (49)$$

7. COMPARISON WITH "INFINITE NUMBER OF BLADES"

A most important characteristic of the results thus far attained is the possibility of comparing them with the relations obtained by the Euler formula. The latter is

$$D = \frac{QY}{\xi} (c_{ua} r_a - c_{ui} r_i)$$

The condition for impact-free entrance in this case is

$$u_{i0} = \omega_o = w_{u_{i0}} = \frac{c}{F_1} \tan \alpha \quad (\text{fig. 7}) \quad (50)$$

where u is the peripheral speed, c the absolute speed, w the relative speed; subscript i denotes inlet, subscript a the exit from the impeller, subscript u the peripheral component, subscript o impact-free inlet.

From equation (50), the value ω_o for impact-free entrance for the case of "infinite number of blades," on the assumption of which the Euler formula is based, is computed as

$$\omega_o = \frac{Q \tan \alpha}{F_1 r_1} = \frac{n q}{r_1^2} \tan \alpha \quad (51)$$

where $F_1 = 2\pi r_1$. The ratio of the angular velocities is

$$\frac{\omega_0'}{\omega_0} = \frac{r_1^2 2\pi (x_b r_b + x_a)}{n \tan \alpha (-N_1' + r_b^2 N_2')} \quad (52)$$

The value r_1 is computed from the transformation function by substituting $z = 0$ and taking the absolute value $r_1 = |\omega_0|$ of the computed value ω_0 . It is immediately found by comparing the two equations (48) and (51) that $\omega_0' \neq \omega_0$ but $\omega_0' < \omega_0$. This results from the following consideration (figs. 8 and 9). In order that torque may be developed, there must exist about each blade a circulation Γ_s which reduces the velocities at the lower side of the blade and increases them at the upper side. As a result, the radial absolute velocity c_0 at the source in the axis is rotated by a certain angle to the right and carried over to c_0' . In order that the water shall enter the blade tangentially - that is, with the velocity ω_0 - the impeller must rotate at the entrance with the peripheral speed u_0' . With the assumption of infinite number of blades $\Gamma_s = 0$, and the absolute velocity is still radially directed at the blade entrance. In this case the tangential approach is affected by the peripheral speed u_0 , and from the diagram (fig. 9) the result is found that $u_0 > u_0'$ or $\omega_0 > \omega_0'$. As the number of blades $n \rightarrow$ approaches infinity $\Gamma_s \rightarrow 0$ and $\lim_{n \rightarrow \infty} \omega_0' = \omega_0$ or

$$\lim_{n \rightarrow \infty} \left[\frac{\omega_0'}{\omega_0} \right] = 1 \quad (53)$$

A comparison of the torque computed by the Euler formula with that computed for the potential flow, is significant only if, in both cases, the same operating condition, determined by ω and q , is assumed. For impact-free entrance, however, a different value of ω was found. For the further computation, only one value - either ω_0 or ω_0' can be used as a basis. For this purpose, ω_0' is chosen because only then are the two condition equations (46) and (47) satisfied if $m = 0$. It would be impractical to start from one of these values as a basis because in one case the fundamental assumptions for the computation would have to be changed, and in the other case the computation would hold only for a pump with guide

vanes. For $\omega = \omega_0'$, the computation by Euler's method no longer gives impact-free entrance. This, however, does not affect the value for the torque since, for the latter, only the rotation at the exit comes into question, ($c_{ui} r_i = 0$)

$$D_o = \frac{QY}{g} c_{ua} r_a, \quad r_a = 1, \quad c_{ua} = u_a - w_{ua} = \omega_0' r_a \tan \alpha,$$

$$D_o' = \frac{QY}{g} n O_o' \quad (54)$$

$$\frac{D_o'}{D_o} = \frac{n O_o'}{c_{ua}} = \frac{n(x_b - x_a) (N_1' - r_b N_2')}{y_a [(x_b r_b + x_a) 2\pi - n \tan \alpha (N_1' + r_b^2 N_2')]} \quad (55)$$

8. VARIABLE NUMBER OF BLADES

Through the choice of the magnitudes α , x_a , and x_b in the transformation function, impellers are obtained whose blades have a certain overlap ratio n . The latter is independent of the number of blades n but the entrance radius r_i varies as a function of n in such a manner that $\lim_{n \rightarrow \infty} r_i = r_a = 1$. The limiting case for the blade

arrangement in the impeller is therefore that of infinitely many infinitesimally small blades. Hence, also

$\lim_{n \rightarrow \infty} \left[\frac{D_o'}{D_o} \right]$ is not equal to 1, but has a different value which will be computed below. From equations (29), (35), and (43)

$$n N_1' = \frac{k}{w} \cos \alpha \int_{-\infty}^{+\infty} M(l)^{1/n} \frac{l^2 dl}{r(l)}$$

$$n N_2' = \frac{k}{w} \cos \alpha \int_{-\infty}^{+\infty} M(l)^{1/n} \frac{dl}{r(l)}$$

where

$$r(l) = [(l - x_a)^2 + y_a^2] [(l - x_b)^2 + y_b^2] \quad (56)$$

$c_n(l, n) = k M(l)^{1/n} l/r(l)$ as a function of n has a physical meaning only for integral positive values of n . It is immediately seen, however, that $c_n(l, n)$ exists also for any arbitrary positive value of n . Denoting by M_{\min} and M_{\max} , respectively, the smallest and largest values that $M(l)$ can assume, then for each arbitrary finite interval $l_u < l < l_o$, we have:

$$M_{\min}^{1/n} \int_{l_u}^{l_o} \frac{l^2 dl}{r(l)} < \int_{l_u}^{l_o} M(l)^{1/n} \frac{l^2 dl}{r(l)} < M_{\max}^{1/n} \int_{l_u}^{l_o} \frac{l^2 dl}{r(l)}$$

or

$$\begin{aligned} M_{\min}^{1/n} \left[\int_{-\infty}^{+\infty} \frac{l^2 dl}{r(l)} - \epsilon_1 \right] &< \int_{-\infty}^{+\infty} M(l)^{1/n} \frac{l^2 dl}{r(l)} \\ &- \epsilon_2 < M_{\max}^{1/n} \left[\int_{-\infty}^{+\infty} \frac{l^2 dl}{r(l)} - \epsilon_1 \right] \end{aligned}$$

This inequality is satisfied for every small $\epsilon_{1,2}$, provided only that $|l_u|$ and $|l_o|$ are chosen sufficiently large. In the limiting case, we have, therefore:

$$M_{\min}^{1/n} \int_{-\infty}^{+\infty} \frac{l^2 dl}{r(l)} \leq \int_{-\infty}^{+\infty} M(l)^{1/n} \frac{l^2 dl}{r(l)} \leq M_{\max}^{1/n} \int_{-\infty}^{+\infty} \frac{l^2 dl}{r(l)}$$

With the aid of equation (32)

$$\lim_{n \rightarrow \infty} \left[\frac{n N_1' \omega}{k \cos \alpha} \right] = \int_{-\infty}^{+\infty} \frac{l^2 dl}{r(l)} = J_1 \quad (57)$$

$$\lim_{n \rightarrow \infty} \left[\frac{n N_2' \omega}{k \cos \alpha} \right] = \int_{-\infty}^{+\infty} \frac{dl}{r(l)} = J_2 \quad (58)$$

The two integrals on the right are to be taken over a fractional rational function and therefore can be evaluated in finite form. The solution is obtained by breaking up into partial fractions

$$\frac{l^2}{(l^2 - 2x_a l + 1)(l^2 - 2x_b l + r_b^2)} = \frac{Al + B}{l^2 - 2x_a l + 1} + \frac{Cl + D}{l^2 - 2x_b l + r_b^2}$$

For the determination of the coefficients, there are four equations:

$$\begin{aligned} + A & + C & & = 0 \\ - 2x_b A + B - 2x_a C + D & = 1 \\ + r_b^2 A - 2x_b B + C - 2x_a D & = 0 \\ + r_b^2 B & + D & = 0 \end{aligned}$$

The determinant of the denominator is:

$$N = \begin{vmatrix} 1 & 0 & 1 & 0 \\ -2x_b & 1 & -2x_a & 1 \\ r_b^2 & -2x_b & 1 & -2x_a \\ 0 & r_b^2 & 0 & 1 \end{vmatrix} = r_b^2 \begin{vmatrix} -2x_a & 1 \\ 1 & -2x_a \end{vmatrix} + \begin{vmatrix} 1 & -2x_a \\ -2x_b & 1 \end{vmatrix}$$

$$-r_b^2 \begin{vmatrix} -2x_b & 1 \\ r_b^2 & -2x_a \end{vmatrix} + \begin{vmatrix} -2x_b & 1 \\ r_b^2 & -2x_b \end{vmatrix} = (r_b^2 - 1)^2 - 4x_a x_b (r_b^2 + 1)$$

$$A = \frac{\begin{vmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & -2x_a & 1 \\ 0 & -2x_b & 1 & -2x_a \\ 0 & r_b^2 & 0 & 1 \end{vmatrix}}{N} = \frac{-2x_b + 2x_a r_b^2}{N}, \quad B = \frac{1 - r_b^2}{N}$$

$$C = -A = \frac{+2x_b - 2x_a r_b^2}{N}, \quad D = \frac{r_b^4 - r_b^2}{N}$$

Having determined the coefficients, the integral can be evaluated (Hütte, Bd. I, or Bierens de Haan, Intégrales Définies, table 22, 14).

$$\int_{-\infty}^{+\infty} \frac{(Al+B) dl}{l^2 - 2x_a l + 1} = \frac{\pi}{y_a} (B + Ax_a); \quad \int_{-\infty}^{+\infty} \frac{(Cl+D) dl}{l^2 - 2x_b l + r_b^2} = \frac{\pi}{y_b} (D + Cx_b)$$

$$J_1 = \int_{-\infty}^{+\infty} \frac{l^2 dl}{r(l)} = \pi \left[\frac{1}{y_a} (B + Ax_a) + \frac{1}{y_b} (D + Cx_b) \right]$$

In exactly the same way the second integral is evaluated:

$$\frac{1}{(l^2 - 2x_a l + 1)(l^2 - 2x_b l + r_b^2)} = \frac{El + F}{l^2 - 2x_a l + 1} + \frac{Gl + H}{l^2 - 2x_b l + r_b^2}$$

$$E = \frac{2x_b - 2x_a}{N}, \quad F = \frac{4x_a^2 - 1 - 4x_a x_b + r_b^2}{N}$$

$$G = -E = \frac{2x_a - 2x_b}{N}, \quad H = \frac{1 - 4x_a x_b + 4x_b^2 - r_b^2}{N}$$

$$J_2 = \pi \left[\frac{1}{y_a} (F + Ex_a) + \frac{1}{y_b} (H + Gx_b) \right]$$

$$\frac{J_1}{J_2} = \frac{(1 - r_b^2 - 2x_a x_b + 2x_a^2 r_b^2)y_b + (r_b^4 - r_b^2 + 2x_b^2 - 2x_a x_b r_b^2)y_a}{(2x_a^2 - 1 + r_b^2 - 2x_a x_b)y_b + (1 - r_b^2 - 2x_a x_b + 2x_b^2)y_a}$$

The fraction is transformed by substituting $y_b = r_b y_a$ and, in the upper right parenthesis, $x_b^2 = x_a^2 r_b^2$. The terms at the lower left and upper right then mutually cancel and there is obtained:

$$I_1 : I_2 = r_b \quad (59)$$

$$\left[\frac{nO_0}{c_{ua}} \right]_{n \rightarrow \infty} = \frac{(x_b - x_a) (J_1 - r_b J_2)}{y_a [2\pi(x_b r_b + x_a) - n \tan \alpha (-J_1 + r_b^2 J_2)]} = \frac{0}{0} \quad (60)$$

The numerator becomes zero on account of equation (59), and the denominator on account of equations (52), (53).

$$\frac{nO_o'}{c_{ua}} = \frac{k_1 \left(\int M(l)^{1/n} \frac{l^2 dl}{r(l)} - r_b \int M(l)^{1/n} \frac{dl}{r(l)} \right)}{k_2 - k_3 \left(- \int M(l)^{1/n} \frac{l^2 dl}{r(l)} + r_b^2 \int M(l)^{1/n} \frac{dl}{r(l)} \right)} = \frac{Z(n)}{N(n)}$$

The functions under the integral signs $M(l)^{1/n} \frac{l^2}{r(l)}$ and $M(l)^{1/n} \frac{1}{r(l)}$ satisfy the following conditions: The functions themselves as well as their derivatives with respect to n are continuous for arbitrary l and positive n , and possess a finite improper integral between the limits $l = -\infty$ and $l = +\infty$. The integrals J_1 and J_2 may therefore be differentiated under the integral sign and there is then obtained:

$$\begin{aligned} \frac{Z'(n)}{N'(n)} &= \\ &= \frac{-k_1 \frac{1}{n^2} \left(\int M(l)^{\frac{1}{n}} \ln M(l) \frac{l^2 dl}{r(l)} - r_b \int M(l)^{\frac{1}{n}} \ln M(l) \frac{dl}{r(l)} \right)}{+k_3 \frac{1}{n^2} \left(- \int M(l)^{\frac{1}{n}} \ln M(l) \frac{l^2 dl}{r(l)} + r_b^2 \int M(l)^{\frac{1}{n}} \ln M(l) \frac{dl}{r(l)} \right)} \end{aligned}$$

On the basis of the same considerations that lead to equations (57) and (58), we have:

$$\lim_{n \rightarrow \infty} \left[\frac{nO_o'}{c_{ua}} \right] = - \frac{k_1 \left(\int \ln M(l) \frac{l^2 dl}{r(l)} - r_b \int \ln M(l) \frac{dl}{r(l)} \right)}{k_3 \left(- \int \ln M(l) \frac{l^2 dl}{r(l)} + r_b^2 \int \ln M(l) \frac{dl}{r(l)} \right)} \quad (61)$$

These integrals may be numerically evaluated, and the limiting value directly computed. The constants k_1 and k_3 are determined by equation (55).

9. CARRYING THROUGH OF A NUMERICAL EXAMPLE

As a practical application of the procedure, let there be chosen an impeller as nearly similar as possible to one in actual use (fig. 13). The magnitudes that determine its shape are the following: Number of blades $n = 6$, blade angle $\alpha = 60^\circ$, overlap ratio $m = 1.31$. The three magnitudes give a ratio between the radii $r_1 : r_2 = 0.455$. By varying the number of blades with α and m constant, impellers are obtained as shown in figures 10 to 15. Of these impellers, naturally only those with $n = 4$ to $n = 12$ are practically useful. In the other cases the ratios of the radii have extreme values. The two magnitudes α and m , determine the as-yet-free constants z_a and z_b of the transformation function, equation (18). If it is desired, with constant ratio of radii, to change the number of blades, it is necessary to change also m and hence also, z_a and z_b ; i.e., a different transformation function is obtained. Since only one transformation function will be computed as an example, only the impellers shown in the figures can be investigated.

There is given $m = 1.31 : \alpha = 60^\circ$. Equation (17):

$$\frac{l}{h} = \frac{2n}{\sin 2\alpha} = 3.03$$

The proportions of the blade system are thus determined. Equation (12) is transcendental and cannot be solved for θ_b . By trial there is found $\theta_b = 30^\circ 8'$

Equation (14) then gives directly $r_b = |z_b| = 372.6$ if $r_a = |z_a| = 1$. It is surprising that the point $z_b = r_b e^{i\theta_b}$ lies so far from the origin. This is connected with the choice of m . The greater the overlap ratio, the farther out does the point z_b move and the more acute is the triangle $(0, z_a, z_b)$.

$$z_a = 1 e^{i(\pi - \theta_b)} = x_a + iy_a; \quad z_b = 372.6 e^{i\theta_b} = x_b + iy_b$$

$$x_a = -0.865, \quad y_a = 0.502; \quad x_b = 322.3, \quad y_b = 187.1$$

The individual factors of the normal velocity $c_n(l)$ can

now be computed, equations (24), (25), (29). The functions $c_n'(l)l$ and $c_n'(l)1/l$ are also known and may be drawn. The integrals of such complicated functions, naturally cannot be expressed in finite form, nor can the definite integrals from $-\infty$ to $+\infty$ be exactly determined. There remains practically only the possibility of evaluating the integrals N_1' and N_2' which occur in the two equations of condition (46) and (47), by planimetry. The difficulty that both integrals are improper is met by an approximation computation. Using equation (31), we may write

$$c_n'(l) = \frac{1}{n} \frac{l}{r(l)} [1 \pm \epsilon'(l)], \text{ if } l \geq l_0' \text{ and } l_0' = f(\epsilon') \quad (62)$$

In the above equation the absolute value of the number $\epsilon'(l)$ can be made smaller than any positive number ϵ by choosing l_0' sufficiently large. Similarly, we may write

$$c_n'(l) = \frac{1}{n} \frac{k}{l^3} [1 + \epsilon(l)] \text{ and } \epsilon(l) < \epsilon, \text{ if } l \geq l_0 \text{ and } l_0 = f(\epsilon) \quad (63)$$

Corresponding equations hold with the same ϵ and l_0 for the functions $c_n'(l)l$ and $\frac{c_n'(l)}{l}$. We then also have

$$\int_{l_0}^{+\infty} c_n'(l)l \, dl = \frac{k}{n} \int_{-l_0}^{+\infty} \frac{dl}{l^3} [1 + \epsilon(l)] = \frac{k}{n} \frac{1}{l_0} (1 + \sigma) \quad (64)$$

where σ is a mean value of the function $\epsilon(l)$ whence follows that $|\sigma| < \epsilon$,

$$\int_{l_0}^{+\infty} \frac{c_n'(l)}{l} \, dl = \frac{k}{n} \int_{l_0}^{+\infty} \frac{dl}{l^4} [1 + \epsilon(l)] = \frac{k}{3n} \frac{1}{l_0^3} (1 + \tau) \quad (65)$$

where $\tau < \epsilon$.

The residual integrals from l_0 to ∞ and from $-l_0$ to $-\infty$, can therefore, to any desired accuracy, be replaced by the above closed expressions. The functions

$c_n'(l)$ and $\frac{c_n'(l)}{l}$ were computed for $-1000 \leq l \leq +1000$ by the exact equation (22), then in the interval $1000 < l < l_0$ by equation (62). The value of l_0 was variously chosen for the different computations, ϵ' being neglected. The residual integrals were replaced by equations (64) and (65), where again σ and τ were neglected. ϵ , σ , τ can easily be estimated so that the integrals may be evaluated to an error of less than 1 percent. The numerical results are collected in the table below, and the functions $\frac{\omega_0'}{\omega_0} = f_1(n)$ and $\frac{nO_0'}{c_{ua}} = f_2(n)$ are plotted in figures 16 and 17.

n	N_1'	N_2'	r_1	$\frac{\omega_0'}{\omega_0}$	$\frac{D_0'}{D_0} = \frac{nO_0'}{c_{ua}}$
1	445	0.01445	0.00851	0.020	+0.377
2	310	.0569	.0923	.234	+.514
3	242	.1025	.2042	.433	+.579
6	150.7	.152	.4545	.715	+.679
12	83.8	.1447	.672	.816	+.685
24	44.9	.0957	.8198	.92	+.704

The values ω_0' for an arbitrary number of blades may be computed to the accuracy obtainable with the planimeter. The computation of O_0' , however, becomes inaccurate for a high number of blades because O_0' is computed as a difference, and for $n \rightarrow \infty$ itself approaches zero. Thus, for example, for $n = 24$ $O_0' = -1.723 + 2.17 \approx +0.45$. The figure 2.17, is obtained by planimentering and contains the unavoidable inaccuracy. The error made in planimentering, as the figures show, is approximately quintupled. This is mainly the result of the extreme ratio of radii $r_1/r_2 = 0.82$. (Also the value $c_{ua} = u_a - w_{ua}$ is determined as a difference.) In actual pumps, such values of the ratio of radii do not occur and an accuracy

of 2 percent may be attained with relative ease. The limiting value for nO_o'/c_{ua} for infinite n is from equation (60) found to be

$$\lim_{n \rightarrow \infty} \left[\frac{nO_o'}{c_{ua}} \right] = 0.775$$

the required integrals being evaluated by the planimeter. Figure 17 shows the result, familiar in pump design, that the torque of a pump is always smaller than the value computed by the Euler method. Only experiment can decide how far the theoretically computed values agree with those empirically obtained. Of the assumptions made here, that of frictionless fluid is of the greatest importance. It makes necessary the assumption of an initially given circulation since no vortices or circulation can arise in a frictionless fluid under any circumstance. Practically, vortices do arise which are shed from the blade tips and carried along with the fluid. Many tests, nevertheless, have shown that the flows in centrifugal machines in the normal operating condition do approach potential flows very closely. (See, for example, the dissertation of Oertli, Zurich, and also the older works of Schuster and Ellon in the *Forschungsheften des V.D.I.*, nos. 82 and 102.) The normal operating condition for every pump is characterized by a definite ratio of rate of discharge to rotational speed. If this ratio is strongly varied - i.e., for extreme operating conditions - flows are obtained that no longer agree with those obtained by the computation. Practically no infinite velocities can arise either at the entrance or the exit of the impeller, even with no impact-free entrance and tangential exit. Moreover, the computation does not predict how far the tangential flow is maintained. Information on this can be obtained only through physical investigations (analogous to airfoil investigations of airplanes). Thus, there are limits to the agreement between actual flow and that determined by computation. The determination of these limits is not essential, however, if the investigation is restricted to the normal operating condition, and since it is this latter condition that is practically of greatest interest, the above computations are of some importance.

Translation by S. Reiss,
National Advisory Committee
for Aeronautics.

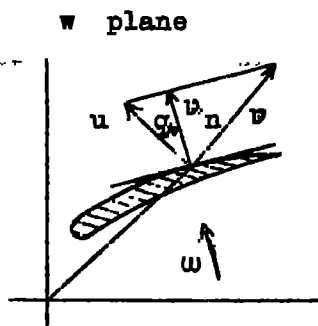


Figure 1.

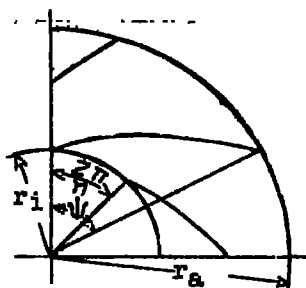


Figure 2.

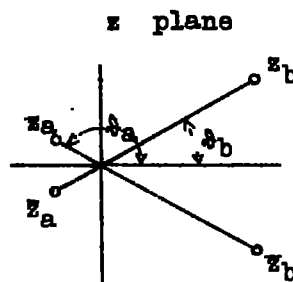


Figure 3.

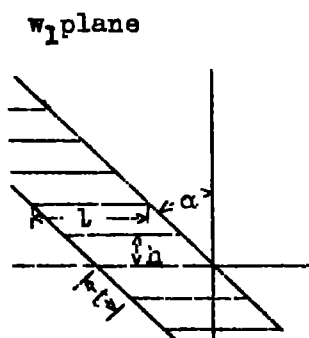


Figure 4

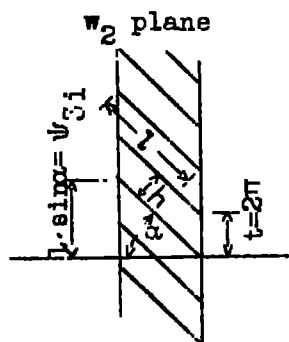


Figure 5

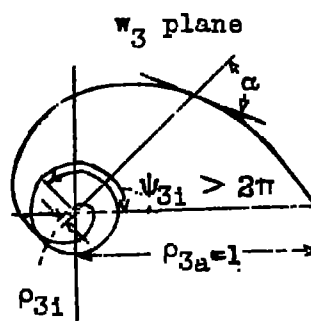


Figure 6

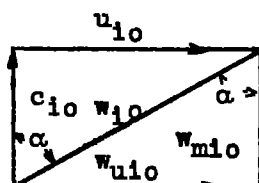


Figure 7

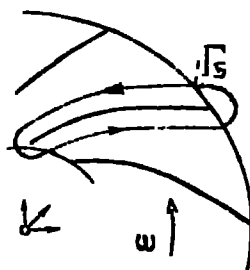


Figure 8

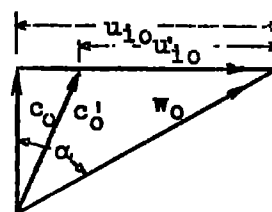


Figure 9

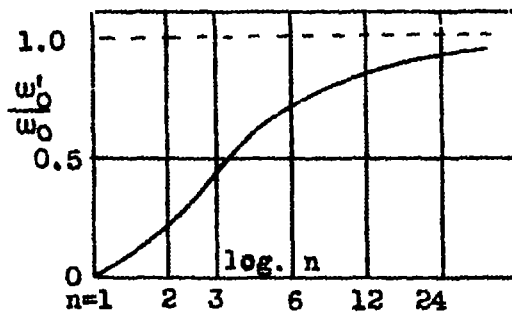
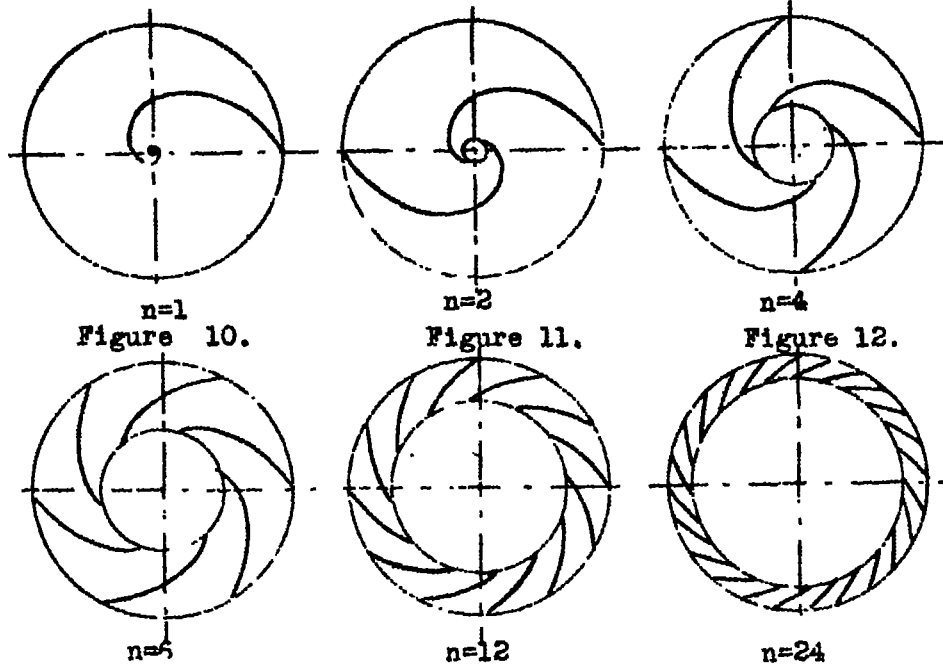


Figure 16

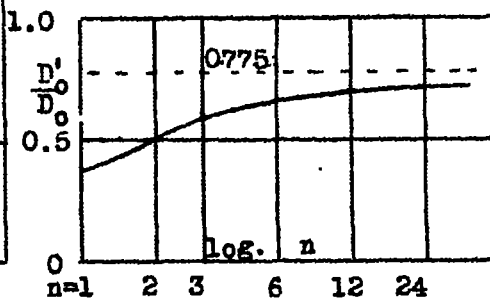


Figure 17

NASA Technical Library



3 1176 01440 7218